Solution to Library/maCalcDB/setVectorCalculus3/ur_vc_13_11.pg

For a sphere of any radius $\rho > 0$, the unit normal vector field is

$$\vec{n}(x, y, z) = \frac{1}{r(x, y, z)}\,\vec{r}(x, y, z)$$

where

$$\vec{r}(x, y, z) = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k},$$
$$r(x, y, z) = \left\|\vec{r}(x, y, z)\right\| = \sqrt{x^2 + y^2 + z^2}.$$

We write just $\vec{n}$, r, and $\vec{r}$ for short, so that

$$\vec{n} = \frac{1}{r}\,\vec{r}.$$

And the sphere has surface area

$$A(\rho) = 4\pi\,\rho^2.$$

(a) The force field $\vec{F} = \vec{F}(x, y, z)$ here is a radial inverse-square force, which means it has the form

$$\vec{F} = \frac{k}{r^2}\,\vec{r}.$$

On the sphere $S(\rho)$ of radius ρ, we have $r = \rho$ and so

$$\vec{F}\cdot\vec{n} = \left(\frac{k}{r^2}\,\vec{r}\right)\cdot\frac{1}{r}\,\vec{r} = \frac{k}{r}\,\frac{1}{r^2}\left(\vec{r}\cdot\vec{r}\right) = \frac{k}{r}\,\frac{1}{r^2}\,r^2 = \frac{k}{r} = \frac{k}{\rho}.$$

Then over such a sphere,

$$\iint_{S(\rho)} \vec{F}\cdot d\,\vec{S} = \iint_{S(\rho)} \left(\vec{F}\cdot\vec{n}\right) d\,S = \iint_{S(\rho)} \frac{k}{\rho}\,d\,S = \frac{k}{\rho}\,A(\rho) = \frac{k}{\rho}\left(4\pi\,\rho^2\right) = 4\pi\,k\,\rho.$$

We are given

$$\iint_{S(a)} \vec{F}\cdot d\,\vec{S} = b,$$

which means

$$4\pi\,k\,a = b$$

whence

$$k = \frac{b}{4\pi\,a}.$$

Hence on the sphere of radius $d = a\,c$,

$$\iint_{S(d)} \vec{F}\cdot d\,\vec{S} = 4\pi\,k\,(d) = 4\pi\,\left(\frac{b}{4\pi\,a}\right)(a\,c) = b\,c.$$

(b) The force field $\vec{F} = \vec{F}(x, y, z)$ here is a radial inverse-cube force, which means it has the form

$$\vec{F} = \frac{k}{r^3}\,\vec{r}.$$

On the sphere $S(\rho)$ of radius ρ, we have $r = \rho$ and so

$$\vec{F}\cdot\vec{n} = \left(\frac{k}{r^3}\,\vec{r}\right)\cdot\frac{1}{r}\,\vec{r} = \frac{k}{r^2}\,\frac{1}{r^2}\,(\vec{r}\cdot\vec{r}) = \frac{k}{r^2}\,\frac{1}{r^2}\,r^2 = \frac{k}{r^2} = \frac{k}{\rho^2}.$$

Then over such a sphere,

$$\iint_{S(\rho)} \vec{F}\cdot d\vec{S} = \iint_{S(\rho)} (\vec{F}\cdot\vec{n})\,dS = \iint_{S(\rho)} \frac{k}{\rho^2}\,dS = \frac{k}{\rho^2}\,A(\rho) = \frac{k}{\rho^2}\left(4\pi\,\rho^2\right) = 4\pi\,k,$$

which is independent of ρ!. We are given

$$\iint_{S(a)} \vec{F}\cdot d\vec{S} = b,$$

which means

$$4\pi\,k = b$$

whence

$$k = \frac{b}{4\pi}.$$

Hence on the sphere of radius $d = a\,c$,

$$\iint_{S(d)} \vec{F}\cdot d\vec{S} = 4\pi\,k\,(d) = 4\pi\,\left(\frac{b}{4\pi}\right)(a\,c) = a\,b\,c.$$